Thus the work done by a constant force $\mathbf{F}$ is the dot product $\mathbf{F}\cdot\mathbf{D}$, where $\mathbf{D}$ is the displacement vector.

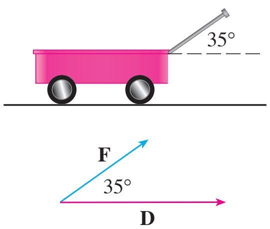


FIGURE 7

EXAMPLE 7 A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

SOLUTION If $\mathbf{F}$ and $\mathbf{D}$ are the force and displacement vectors, as pictured in Figure 7, then the work done is

$$\begin{aligned} W &= \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos 35^\circ \\ &= (70)(100)\cos 35^\circ \approx 5734\ \text{N}\cdot\text{m} = 5734\ \text{J} \end{aligned}$$

EXAMPLE 8 A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

SOLUTION The displacement vector is $\mathbf{D} = \overrightarrow{PQ} = \langle 2, 5, 2\rangle$, so by Equation 12, the work done is

$$\begin{aligned} W &= \mathbf{F}\cdot\mathbf{D} = \langle 3, 4, 5\rangle \cdot \langle 2, 5, 2\rangle \\ &= 6 + 20 + 10 = 36 \end{aligned}$$

If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 joules.

12.3 EXERCISES

1. Which of the following expressions are meaningful? Which are meaningless? Explain.
(a) $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ (b) $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$
(c) $|\mathbf{a}|(\mathbf{b}\cdot\mathbf{c})$ (d) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$
(e) $\mathbf{a}\cdot\mathbf{b}+\mathbf{c}$ (f) $|\mathbf{a}|\cdot(\mathbf{b}+\mathbf{c})$

2. Find the dot product of two vectors if their lengths are 6 and $\frac{1}{3}$ and the angle between them is $\pi/4$.

3–10 Find $\mathbf{a}\cdot\mathbf{b}$.

3. $\mathbf{a} = \langle -2, \frac{1}{3}\rangle$, $\mathbf{b} = \langle -5, 12\rangle$

4. $\mathbf{a} = \langle -2, 3\rangle$, $\mathbf{b} = \langle 0.7, 1.2\rangle$

5. $\mathbf{a} = \langle 4, 1, \frac{1}{4}\rangle$, $\mathbf{b} = \langle 6, -3, -8\rangle$

6. $\mathbf{a} = \langle s, 2s, 3s\rangle$, $\mathbf{b} = \langle t, -t, 5t\rangle$

7. $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 9\mathbf{k}$

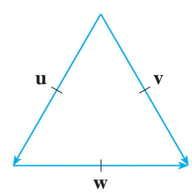
8. $\mathbf{a} = 4\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

9. $|\mathbf{a}| = 6$, $|\mathbf{b}| = 5$, the angle between $\mathbf{a}$ and $\mathbf{b}$ is $2\pi/3$

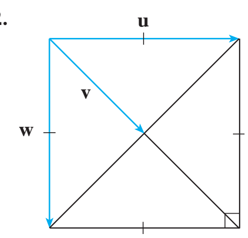
10. $|\mathbf{a}| = 3$, $|\mathbf{b}| = \sqrt{6}$, the angle between $\mathbf{a}$ and $\mathbf{b}$ is 45°

11–12 If $\mathbf{u}$ is a unit vector, find $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\cdot\mathbf{w}$.

11.



12.



13. (a) Show that $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$.
(b) Show that $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$.

14. A street vendor sells a hamburgers, b hot dogs, and c soft drinks on a given day. He charges \$2 for a hamburger, \$1.50 for a hot dog, and \$1 for a soft drink. If $\mathbf{A} = \langle a, b, c\rangle$ and $\mathbf{P} = \langle 2, 1.5, 1\rangle$, what is the meaning of the dot product $\mathbf{A}\cdot\mathbf{P}$?

15–20 Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

15. $\mathbf{a} = \langle -8, 6\rangle$, $\mathbf{b} = \langle \sqrt{7}, 3\rangle$

16. $\mathbf{a} = \langle \sqrt{3}, 1\rangle$, $\mathbf{b} = \langle 0, 5\rangle$

17. $\mathbf{a} = \langle 3, -1, 5 \rangle, \quad \mathbf{b} = \langle -2, 4, 3 \rangle$

18. $\mathbf{a} = \langle 4, 0, 2 \rangle, \quad \mathbf{b} = \langle 2, -1, 0 \rangle$

19. $\mathbf{a} = \mathbf{j} + \mathbf{k}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

20. $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}, \quad \mathbf{b} = 4\mathbf{i} - 3\mathbf{k}$

21–22 Find, correct to the nearest degree, the three angles of the triangle with the given vertices.

21. $A(1, 0), \quad B(3, 6), \quad C(-1, 4)$

22. $D(0, 1, 1), \quad E(-2, 4, 3), \quad F(1, 2, -1)$

23–24 Determine whether the given vectors are orthogonal, parallel, or neither.

23. (a) $\mathbf{a} = \langle -5, 3, 7 \rangle, \quad \mathbf{b} = \langle 6, -8, 2 \rangle$
(b) $\mathbf{a} = \langle 4, 6 \rangle, \quad \mathbf{b} = \langle -3, 2 \rangle$
(c) $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$
(d) $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}, \quad \mathbf{b} = -3\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$

24. (a) $\mathbf{u} = \langle -3, 9, 6 \rangle, \quad \mathbf{v} = \langle 4, -12, -8 \rangle$
(b) $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$
(c) $\mathbf{u} = \langle a, b, c \rangle, \quad \mathbf{v} = \langle -b, a, 0 \rangle$

25. Use vectors to decide whether the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, and $R(6, -2, -5)$ is right-angled.

26. For what values of b are the vectors $\langle -6, b, 2 \rangle$ and $\langle b, b^2, b \rangle$ orthogonal?

27. Find a unit vector that is orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$.

28. Find two unit vectors that make an angle of $60°$ with $\mathbf{v} = \langle 3, 4 \rangle$.

29–33 Find the direction cosines and direction angles of the vector. (Give the direction angles correct to the nearest degree.)

29. $\langle 3, 4, 5 \rangle$

30. $\langle 1, -2, -1 \rangle$

31. $2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$

32. $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

33. $\langle c, c, c \rangle, \quad$ where $c > 0$

34. If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$, find the third direction angle γ.

35–40 Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$.

35. $\mathbf{a} = \langle 3, -4 \rangle, \quad \mathbf{b} = \langle 5, 0 \rangle$

36. $\mathbf{a} = \langle 1, 2 \rangle, \quad \mathbf{b} = \langle -4, 1 \rangle$

37. $\mathbf{a} = \langle 3, 6, -2 \rangle, \quad \mathbf{b} = \langle 1, 2, 3 \rangle$

38. $\mathbf{a} = \langle -2, 3, -6 \rangle, \quad \mathbf{b} = \langle 5, -1, 4 \rangle$

39. $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}, \quad \mathbf{b} = \mathbf{j} + \frac{1}{2}\mathbf{k}$

40. $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

41. Show that the vector $\operatorname{orth}_{\mathbf{a}} \mathbf{b} = \mathbf{b} - \operatorname{proj}_{\mathbf{a}} \mathbf{b}$ is orthogonal to $\mathbf{a}$. (It is called an **orthogonal projection** of $\mathbf{b}$.)

42. For the vectors in Exercise 36, find $\operatorname{orth}_{\mathbf{a}} \mathbf{b}$ and illustrate by drawing the vectors $\mathbf{a}$, $\mathbf{b}$, $\operatorname{proj}_{\mathbf{a}} \mathbf{b}$, and $\operatorname{orth}_{\mathbf{a}} \mathbf{b}$.

43. If $\mathbf{a} = \langle 3, 0, -1 \rangle$, find a vector $\mathbf{b}$ such that $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = 2$.

44. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors.
(a) Under what circumstances is $\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \operatorname{comp}_{\mathbf{b}} \mathbf{a}$?
(b) Under what circumstances is $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \operatorname{proj}_{\mathbf{b}} \mathbf{a}$?

45. Find the work done by a force $\mathbf{F} = 8\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$ that moves an object from the point $(0, 10, 8)$ to the point $(6, 12, 20)$ along a straight line. The distance is measured in meters and the force in newtons.

46. A tow truck drags a stalled car along a road. The chain makes an angle of $30°$ with the road and the tension in the chain is 1500 N. How much work is done by the truck in pulling the car 1 km?

47. A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of $40°$ above the horizontal moves the sled 80 ft. Find the work done by the force.

48. A boat sails south with the help of a wind blowing in the direction S36°E with magnitude 400 lb. Find the work done by the wind as the boat moves 120 ft.

49. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

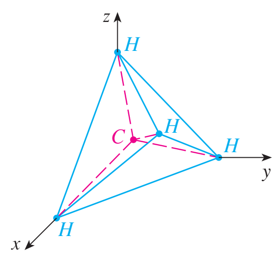
Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

50. If $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a sphere, and find its center and radius.

51. Find the angle between a diagonal of a cube and one of its edges.

52. Find the angle between a diagonal of a cube and a diagonal of one of its faces.

53. A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The *bond angle* is the angle formed by the H—C—H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about $109.5°$. [*Hint:* Take the vertices of the tetrahedron to be the points $(1, 0, 0)$, $(0, 1, 0)$, (0, 0, 1), and (1, 1, 1) as shown in the figure. Then the centroid is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.]



54. If $\mathbf{c} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are all nonzero vectors, show that $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

55. Prove Properties 2, 4, and 5 of the dot product (Theorem 2).

56. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

57. Use Theorem 3 to prove the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leqslant |\mathbf{a}||\mathbf{b}|$$

58. The Triangle Inequality for vectors is

$$|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$$

(a) Give a geometric interpretation of the Triangle Inequality.
(b) Use the Cauchy-Schwarz Inequality from Exercise 57 to prove the Triangle Inequality. [*Hint:* Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and use Property 3 of the dot product.]

59. The Parallelogram Law states that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

(a) Give a geometric interpretation of the Parallelogram Law.
(b) Prove the Parallelogram Law. (See the hint in Exercise 58.)

60. Show that if $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal, then the vectors $\mathbf{u}$ and $\mathbf{v}$ must have the same length.

12.4 THE CROSS PRODUCT

The **cross product** $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$, unlike the dot product, is a vector. For this reason it is also called the **vector product**. Note that $\mathbf{a} \times \mathbf{b}$ is defined only when $\mathbf{a}$ and $\mathbf{b}$ are *three-dimensional* vectors.

1 DEFINITION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

This may seem like a strange way of defining a product. The reason for the particular form of Definition 1 is that the cross product defined in this way has many useful properties, as we will soon see. In particular, we will show that the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

In order to make Definition 1 easier to remember, we use the notation of determinants. A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,
$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

2
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$